$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|\Delta Y_t^{n+1}\right|^2\\
&+\mathbb{E}\int_t^T \frac{\left(\delta^2-2\right)s^{2H-1}-2}{\delta s^{2H-1}}a^2(s)\\
&\times e^{\delta A(s)}\left|\Delta Y_s^{n+1}\right|^2 ds \qquad (17)\\
&+\frac{2}{M}\mathbb{E}\left(\int_t^T s^{2H-1}e^{\delta A(s)}\left|\Delta Z_s^{n+1}\right|^2 ds\right)\\
\leq\ & 2\delta\mathbb{E}\int_t^T e^{\delta A(s)}\\
&\times\left(a^2(s)\left|\Delta Y_s^n\right|^2 ds+s^{2H-1}\left|\Delta Z_s^n\right|^2\right)ds.
\end{aligned}$$

Choosing $\delta>0$ such that $\left(\delta-\frac{2}{\delta}-\frac{2}{\delta s^{2H-1}}\right)>1$ for any $t\leq s\leq T$ and using the inequality (14), and note that $M>2$, we have

$$\begin{aligned}
&\mathbb{E}\int_{t_k}^T e^{\delta A(s)}\left(a^2(s)\left|\Delta Y_s^{n+1}\right|^2+s^{2H-1}\left|\Delta Z_s^{n+1}\right|^2\right)ds\\
\leq\ &\delta MF\mathbb{E}\int_{t_k}^T a^2(s)e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2 ds\\
&+\delta(MG+2)\mathbb{E}\int_{t_k}^T s^{2H-1}e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2 ds.
\end{aligned}$$

Now choosing $\delta>0$ and taking k large enough such that $\delta MF\leq\frac{1}{4}$ and $\delta(MG+2)\leq\frac{1}{4}$, we deduce

$$\begin{aligned}
&\mathbb{E}\int_{t_k}^T e^{\delta A(s)}a^2(s)\left|\Delta Y_s^{n+1}\right|^2 ds\\
&+\mathbb{E}\int_{t_k}^T e^{\delta A(s)}s^{2H-1}\left|\Delta Z_s^{n+1}\right|^2 ds\\
\leq\ &\frac{1}{2}\mathbb{E}\int_{t_k}^T e^{\delta A(s)}a^2(s)\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2 ds\\
&+\frac{1}{2}\mathbb{E}\int_{t_k}^T e^{\delta A(s)}s^{2H-1}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2 ds.
\end{aligned}$$

As a consequence, we deduce that $(Y^n,Z^n)_{n\geq0}$ is a Cauchy sequence in $\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$. Then there exists a pair $(Y,Z)\in\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$ being a limit of $(Y^n,Z^n)_{n\geq1}$, i.e.

$$\begin{aligned}
&\mathbb{E}\int_0^T a^2(s)e^{\delta A(s)}\left|Y_s^n-Y_s\right|^2 ds\\
&+\mathbb{E}\int_0^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n-Z_s\right|^2 ds\\
\to\ &0,\ \text{as } n\to\infty.
\end{aligned}$$

It remains to show that the pair (Y,Z) satisfies equation (2) on the interval $[0,T]$. We have for any $t\in[t_k,T]$,

$$\begin{aligned}
&Y_t^{n+1}-\xi-\int_t^T f(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n,\acute{Z}_s^n)ds\\
&\underset{n\to\infty}{\to}Y_t-\xi-\int_t^T f(s,\eta_s,Y_s,Z_s,\acute{Y}_s,\acute{Z}_s)ds,
\end{aligned}$$

in $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$. And $Z_t^n\mathbf{1}_{[t,T]}\to Z_t\mathbf{1}_{[t,T]}$ in $\mathbb{L}^2(\Omega,\mathcal{F},\mathcal{H})$. Arguing as in the proof of Theorem 23 in Maticiuc and Nie [11] we show that (Y,Z) satisfies (2) on $[t_k,T]$. The next step is to solve the equation on $[t_{k-1},t_k]$. With the same arguments, repeating the above technique we obtain a uniqueness of the solution of MF-BSDE with respect to fBm on the whole interval $[0,T]$.

Uniqueness part: Let $\left(Y_\cdot^1,Z_\cdot^1\right)$ and $\left(Y_\cdot^2,Z_\cdot^2\right)$ two solutions of fractional MF-BSDEs (2), then by Itô's formula applied to $e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2$ it follows that, $\forall t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2+\delta\mathbb{E}\int_t^T a^2(s)e^{\delta A(s)}\left|Y_s^1-Y_s^2\right|^2 ds\\
=\ &2\mathbb{E}\int_t^T e^{\delta A(s)}\left(Y_s^1-Y_s^2\right)\\
&\times\left(f(s,\eta_s,Y_s^1,Z_s^1,\acute{Y}_s^1,\acute{Z}_s^1)-f(s,\eta_s,Y_s^2,Z_s^2,\acute{Y}_s^2,\acute{Z}_s^2)\right)ds\\
&-2\mathbb{E}\left(\int_t^T e^{\delta A(s)}\mathcal{D}_s^H\left(Y_s^1-Y_s^2\right)\left(Z_s^1-Z_s^2\right)ds\right),
\end{aligned}$$

then, we can write

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2\\
&+\mathbb{E}\int_t^T\left(\delta-4-\frac{2\delta M}{s^{2H-1}}\right)a^2(s)e^{\delta A(s)}\left|Y_s^1-Y_s^2\right|^2 ds\\
&+\frac{2\delta-2}{\delta M}\mathbb{E}\left(\int_t^T e^{\delta A(s)}s^{2H-1}\left|Z_s^1-Z_s^2\right|^2 ds\right)\\
\leq\ &0,
\end{aligned}$$

which can be chosen δ, M such that $\delta-4-\frac{2\delta M}{s^{2H-1}}>0$ for any $t\leq s\leq T$ and $\frac{2\delta-2}{\delta M}>0$. Thus, we deduce that

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2\\
&+\mathbb{E}\int_t^T e^{\delta A(s)}\left(a^2(s)\left|Y_s^1-Y_s^2\right|^2+s^{2H-1}\left|Z_s^1-Z_s^2\right|^2\right)ds\\
\leq\ &0.
\end{aligned}$$

This implies $Y_t^1=Y_t^2$ and $Z_s^1=Z_s^2$. The result follows. □

2.3 Comparison Theorem

In this subsection we study a comparison theorem for the fractional MF-BSDEs of the following form:

$$\begin{cases}-dY_t^i=\mathbb{E}'\left(f^i(t,\eta_t,Y_t^i,Z_t^i,\acute{Y}_t^i,\acute{Z}_t^i)\right)dt-Z_t^i dB_t^H\\ Y_T^i=\xi^i,\quad 0\leq t\leq T.\end{cases}\tag{18}$$

where for any $i\in\{1,2\}$, $f^i:\Omega\times[0,T]\times\mathbb{R}^5\to\mathbb{R}$.

We assume in addition that

$$\textbf{(H1.5)}\begin{cases}\xi^1\leq\xi^2,\\ f^1(s,\eta,y,z,\acute{y},\acute{z})\leq f^2(s,\eta,y,z,\acute{y},\acute{z}),\\ \forall(s,\ \eta,\ y,\ z,\ \acute{y},\ \acute{z})\in[0,T]\times\mathbb{R}^5.\end{cases}$$

We have the following theorem:

Theorem. *Suppose that* $\left(\xi^1,f^1\right)$ *and* $\left(\xi^2,f^2\right)$ *satisfy* ***(H1.1)–(H1.5)***.

If $\left(Y_s^i,Z_s^i\right)$, $i=1,2$ *are solutions to Eq.* (18), *then we have*

$$\forall t\in[0,T],\qquad Y^1\leq Y^2,\qquad \mathbb{P}-a.s.$$

Proof. Let us define $\Delta Y_t = Y_t^2 - Y_t^1$, $\Delta Z_t = Z_t^2 - Z_t^1$, $\Delta\acute{Y}_t = \acute{Y}_t^2 - \acute{Y}_t^1$, $\Delta\acute{Z}_t = \acute{Z}_t^2 - \acute{Z}_t^1$, $\Delta\xi = \xi^2 - \xi^1$ and

$$\begin{aligned} &\Delta f\left(t, \eta_t, \Delta Y_t, \Delta Z_t, \Delta\acute{Y}_t, \Delta\acute{Z}_t\right) \\ =\ & \mathbb{E}' f^2\left(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2\right) \\ & -\mathbb{E}' f^1(t, \eta_t, Y_t^1, Z_t^1, \acute{Y}_t^1, \acute{Z}_t^1). \end{aligned}$$

It follows that $(\Delta Y_t, \Delta Z_t)_{t\in[0,T]}$ satisfies the fractional MF-BSDE for any $0 \le t \le T$

$$\begin{aligned} & \Delta Y_t \\ =\ & \Delta\xi + \int_t^T \Delta f\left(s, \eta_s, \Delta Y_s, \Delta Z_s, \Delta\acute{Y}_s, \Delta\acute{Z}_s\right) ds \\ & -\int_t^T \Delta Z_s dB_s^H. \end{aligned}$$

Applying Itô's formula to $e^{\delta A(t)}\left|\Delta Y_t^-\right|^2$, we obtain

$$\begin{aligned} & \mathbb{E}e^{\delta A(t)}\left|\Delta Y_t^-\right|^2 + \delta\mathbb{E}\int_t^T a^2(s)\, e^{\delta A(s)}\left|\Delta Y_s^-\right|^2 ds \\ & +\frac{2}{M}\mathbb{E}\int_t^T \mathbf{1}_{\{\Delta Y_s<0\}} e^{\delta A(s)} s^{2H-1}\left|\Delta Z_s\right|^2 ds \\ =\ & \mathbb{E}\left(e^{\delta A(T)}\Delta\xi^-\right) - 2\mathbb{E}\int_t^T \mathbf{1}_{\{\Delta Y_s<0\}} e^{\delta A(s)}\Delta Y_s^- \times \\ & \Delta f\left(s, \eta_s, \Delta Y_s, \Delta Z_s, \Delta\acute{Y}_s, \Delta\acute{Z}_s\right) ds. \end{aligned}$$

Since $\mathbb{E}' f^2\left(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2\right) \ge \mathbb{E}' f^1(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2)$ and $\Delta\xi = \xi^1 - \xi^2 \ge 0$, we have

$$\begin{aligned} & \mathbb{E}e^{\delta A(t)}\left|\Delta Y_t^-\right|^2 + \delta\mathbb{E}\int_t^T a^2(s)\, e^{\delta A(s)}\left|\Delta Y_s^-\right|^2 ds \\ & +\frac{2}{M}\mathbb{E}\int_t^T \mathbf{1}_{\{\Delta Y_s<0\}} e^{\delta A(s)} s^{2H-1}\left|\Delta Z_s\right|^2 ds \\ \le\ & 2\mathbb{E}\int_t^T \mathbf{1}_{\{\Delta Y_s<0\}} e^{\delta A(s)}\Delta Y_s^- \\ & \times(\mathbb{E}' f^1\left(s, \eta_s, Y_s^2, Z_s^2, \acute{Y}_s^2, \acute{Z}_s^2\right) \\ & -\mathbb{E}' f^1(s, \eta_s, Y_s^1, Z_s^1, \acute{Y}_s^1, \acute{Z}_s^1))ds. \end{aligned}$$

From $(H1.1)$, $(H1.2)$ and Young's inequality, we have

$$\begin{aligned} & \Delta Y_s^-\mathbb{E}'\left(f^1\left(s, \eta_s, Y_s^2, Z_s^2, \acute{Y}_s^2, \acute{Z}_s^2\right)\right. \\ & \left.-f^1(s, \eta_s, Y_s^1, Z_s^1, \acute{Y}_s^1, \acute{Z}_s^1)\right) \\ \le\ & \frac{1}{2}\left(4 + \frac{4M}{s^{2H-1}}\right) a^2(s)\left|\Delta Y_s^-\right|^2 + \frac{s^{2H-1}}{2M}\left|\Delta Z_s\right|^2. \end{aligned}$$

Finally, it follows that

$$\mathbb{E}\left(e^{\delta A(t)}\left|\Delta Y_t^-\right|^2 + \int_t^T\left(\delta - 4 - \tfrac{4M}{s^{2H-1}}\right) a^2(s)\, e^{\delta A(s)}\left|\Delta Y_s^-\right|^2 ds\right) + \tfrac{1}{M}\mathbb{E}\left(\int_t^T \mathbf{1}_{\{\Delta Y_s<0\}} e^{\delta A(s)} s^{2H-1}\left|\Delta Z_s\right|^2 ds\right) \le 0$$

Therefore, choosing $\delta > 0$ and $M > 0$, such that $\left(\delta - 4 - \frac{4M}{s^{2H-1}}\right) \ge 0$, we derive that $\Delta Y_t^- = 0$ $\mathbb{P}-a.s.$ for all $t \in [0,T]$, which implies that $\Delta Y_t = Y_t^2 - Y_t^1 \ge 0$ $\mathbb{P}-a.s.$ for all $t \in [0,T]$. □

3 Fractional MF-BSDE with Continuous and Stochastic Linear Growth Coefficients.

The objective of this section is to prove an existence theorem for MF-BSDEs (2) with Hurst parameter $H > \frac{1}{2}$ when the coefficient f is continuous with stochastic linear growth. More precisely, the coefficient $f : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is measurable and the terminal value $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T-$measurable satisfying the following assumptions for $\delta > 0$.

(A1) The following hold:

i) For fixed ω and t, $f(t, \omega, x, \cdot, \cdot, \cdot)$ is continuous.

ii) For all $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$

$$|f(t, \omega, x, y, z, \acute{y})| \le \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)(|z|).$$

where φ, r and θ are three nonnegative processes such that for $a.e.$ $t \in [0,T]$, $\varphi(t)$, $r(t)$ and $\theta(t)$ $\mathcal{F}_t-$measurable.

iii) For all $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$, $f(t, \omega, x, y, z, \acute{y})$ is $\mathcal{F}_t-$measurable.

(A2) For any $t \in [0,T]$,

$$\begin{aligned} a^2(t) &= r(t) + \theta^2(t) > 0, \text{ and} \\ A(t) &= \int_0^t a^2(s)\, ds < \infty. \end{aligned}$$

(A3) One has

$$\mathbb{E}e^{\delta A(T)}|\xi|^2 + \mathbb{E}\int_0^T e^{\delta A(t)}\left(\frac{|\varphi(t)|^2}{a^2(t)} + a^2(t)|\eta_t|^2\right) dt < \infty.$$

To reach our objective, we first give the following useful approximation lemma, which generalizes the corresponding result of Lepeltier and San Martin [10].

Lemma. Let $f : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ be a measurable function such that:

For a.s. every $(t, \omega) \in [0,T] \times \Omega$, $f(t, \omega, x, y, z, \acute{y})$ is a continuous.

For every $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$

$$\begin{aligned} & |f(t, \omega, x, y, z, \acute{y})| \\ \le\ & \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)|z|. \end{aligned}$$

where φ, r and θ are three nonnegative processes such that for $a.e.$ $t \in [0,T]$, $\varphi(t)$, $r(t)$ and $\theta(t)$ $\mathcal{F}_t-$measurable.

Then exists the sequence of fonction f_n

$$\begin{aligned} & f_n(t, \omega, x, y, z, \acute{y}) \\ =\ & \inf_{(\tilde{y}, \tilde{z}, \tilde{y}', \tilde{z}')\in\mathbb{Q}}(f\left(t, \omega, x, \tilde{y}, \tilde{z}, \tilde{y}'\right) \\ & +n\left(r(t)\left(|y - \tilde{y}| + \left|\acute{y} - \tilde{y}'\right|\right) + \theta(t)|z - \tilde{z}|\right)), \end{aligned}$$

are well defined for $n \ge 1$ and satisfy the following conditions

(i) For all $n \ge 1$, $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^5$,

$$\begin{aligned} & |f_n(t, \omega, x, y, z, \acute{y})| \\ \le\ & \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)|z|. \end{aligned}$$

(ii) For any $(t,\omega,x,y,z,\acute{y})$, $f_n(t,\omega,x,y,z,\acute{y})$ is non-decreasing in n.

(iii) For all $(t,\omega,x,y,z,\acute{y}) \in [0,T]\times\Omega\times\mathbb{R}^4$, if $(t,\omega,x,y_n,z_n,\acute{y}_n)\to(t,\omega,x,y,z,\acute{y})$, then

$$f_n(t,\omega,x,y_n,z_n,\acute{y}_n)\to f(t,\omega,x,y,z,\acute{y}).$$

(iv) For any $n\geq 1$, $(t,\omega)\in[0,T]\times\Omega$, for all $(t,\omega,x,y,z,\acute{y})\in[0,T]\times\Omega\times\mathbb{R}^4$ and $\left(t,\omega,x,\tilde{y},\tilde{z},\tilde{y}'\right)\in[0,T]\times\Omega\times\mathbb{R}^4$, we have

$$\begin{aligned}&\left|f_n(t,\omega,x,y,z,\acute{y})-f_n\left(t,\omega,x,\tilde{y},\tilde{z},\tilde{y}'\right)\right|\\ \leq\ & n\left(r(t)\left(|y-\tilde{y}|+\left|\acute{y}-\tilde{y}'\right|\right)+\theta(t)|z-\tilde{z}|\right).\end{aligned}$$

3.1 Existence Result

Now, by the approximation method of the function f (**previous lemma**) and the comparison theorem, we establish the following existence theorem.

Theorem. *Assume* ***(A1)-(A3)****. Then, for δ sufficiently large, the MF-BSDE* (2) *has a minimal solution* $\left(\underline{Y},\underline{Z}\right)\in\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$.

Proof. We only prove that fractional MF-BSDE (2) has a minimal solution. Since $(A1)$ holds, it follows from **previous lemma** that there exists a sequence of stochastic Lipschitz-continuous functions f_n associated with f, which is non-decreasing in n.

Since

$$g(t)=\varphi(t)+r(t)(|x|+|y|+|\acute{y}|)+\theta(t)|z|$$

is stochastic Lipschitz, the existence and uniqueness result in the previous section implies that there exists a unique solution $(U,V)\in\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$ for fractional MF-BSDEs with data(ξ,g).

Now, for any $n\geq 1$, let a_n and A_n be two random processes with positive values defined by

$$\begin{aligned}a_n^2(t)&=nr(t)+n^2\theta^2(t)>0,\text{ and}\\ A_n(t)&=\int_0^t a_n^2(s)\,ds<\infty.\end{aligned}$$

Then, in view of $(A1)-(A2)$, $a_n(t)$ and $A_n(t)$ are $\mathcal{F}_t-$measurable, for a.e. $t\in[0,T]$ such that for any $n\geq 1$, $0<a<a_n$ and $A<A_n<n^2A$. Thus, it is clair to deduce that for any $n\geq 1$, $\tilde{\mathcal{V}}^{\frac{1}{2},a_n}_{[0,T]}\subset\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}$. Moreover, from $(A3)$, (ξ,f_n) satisfies the following conditions for any $n\geq 1$:

$$\begin{aligned}\mathbb{E}\left(e^{\delta A_n(T)}|\xi|^2\right)&\leq\mathbb{E}\left(e^{\delta n^2A(T)}|\xi|^2\right)\\ &<\infty,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\int_0^T e^{\delta A_n(t)}\frac{|f_n(t,\omega,x,0,0,0)|^2}{a_n^2(t)}dt\\ \leq\ &\int_0^T e^{\delta n^2A(t)}\frac{|\varphi(t)|^2}{a^2(t)}dt<\infty.\end{aligned}$$

Therefore, we get again from the previous section that for every $n\geq 1$ there exists a unique solution $(Y^n,Z^n)\in\tilde{\mathcal{V}}^{\frac{1}{2},a_n}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$ for fractional MF-BSDE with data:

$$\begin{cases}-dY_t^n=\mathbb{E}'\left(f_n(t,\eta_t,Y_t^n,Z_t^n,\acute{Y}_t^n)\right)dt-Z_t^n dB_t^H,\\ Y_T^n=\xi,\ 0\leq t\leq T.\end{cases}\tag{19}$$

Consequently, for any $n\geq 1$, $(Y^n,Z^n)\in\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$.

On the other hand, since for fixed $(t,\omega,x,y,z,\acute{y})$ and all $n\geq 1$,

$$\begin{aligned}f_n(t,\omega,x,y,z,\acute{y})&\leq f_{n+1}(t,\omega,x,y,z,\acute{y})\\ &\leq\varphi(t)+r(t)(|x|+|y|+|\acute{y}|)+\theta(t)|z|,\end{aligned}$$

it follows from the comparison theorem that for every $n\geq 1$,

$$Y^n\leq Y^{n+1}\leq U,\ d\mathbb{P}\otimes dt-a.s.\tag{20}$$

The idea of the proof is to establish that the limit of the sequence (Y^n,Z^n) is a solution of the fractional MF-BSDE (2). To this end, we will sketch the proof in four steps.

Step 1: A priori estimates. There exists a constant $C>0$ independent of n such that

$$\begin{aligned}&\mathbb{E}e^{\delta A(t)}|Y_t^n|^2+\mathbb{E}\int_t^T e^{\delta A(s)}a^2(s)|Y_s^n|^2ds\\ &+\mathbb{E}\int_t^T e^{\delta A(s)}s^{2H-1}|Z_s^n|^2ds\\ \leq\ & C,\end{aligned}\tag{21}$$

where C is a positive constant which may be different from line to line.

Indeed, for any $\delta>0$, Itô's formula applied to $e^{\delta A(t)}|Y_t^n|^2$ provides

$$\begin{aligned}&e^{\delta A(t)}|Y_t^n|^2\\ =\ &e^{\delta A(T)}|\xi|^2+2\int_t^T e^{\delta A(s)}Y_s^n\\ &\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds\\ &-2\int_t^T e^{\delta A(s)}Y_s^nZ_s^n dB_s^H-2\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}Y_s^nZ_s^n ds\\ &-\delta\int_t^T e^{\delta A(s)}|Y_s^n|^2ds.\end{aligned}$$

Taking expectation, we get

$$\begin{aligned}&\mathbb{E}e^{\delta A(t)}|Y_t^n|^2+\delta\mathbb{E}\int_t^T a^2(s)e^{\delta A(s)}|Y_s^n|^2ds\\ &+2\mathbb{E}\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}Y_s^nZ_s^n ds\\ =\ &\mathbb{E}e^{\delta A(T)}|\xi|^2+2\mathbb{E}\int_t^T e^{\delta A(s)}Y_s^n\\ &\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds.\end{aligned}$$

It is known (see example Hu and Peng, [9], Maticiuc and Nie, [11]) that $\mathcal{D}_t^{\mathcal{H}}Y_t^n=(\hat{\sigma}(t)/\sigma(t))Z_t^n$. Moreover by Remark 6 in Maticiuc and Nie [11], there exists $M>0$ such that for all $t\in[0,T]$, $t^{2H-1}/M\leq\hat{\sigma}(t)/\sigma(t)\leq Mt^{2H-1}$.

Then, we have

$$\begin{aligned}&\mathbb{E}e^{\delta A(t)}\left|Y_t^n\right|^2+\delta\mathbb{E}\int_t^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2 ds\\&+\frac{2}{M}\mathbb{E}\int_t^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2 ds\\\leq\;&\mathbb{E}e^{\delta A(T)}\left|\xi\right|^2+2\mathbb{E}\int_t^T e^{\delta A(s)}Y_s^n\\&\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds,\end{aligned}\tag{22}$$

assumption (A2) and property (iv) from previous lemma together with Young's inequality imply,

$$\begin{aligned}&2Y_s^n\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)\\\leq\;&\left(5+\frac{M}{s^{2H-1}}\right)a^2(s)\left|Y_s^n\right|^2\\&+\frac{s^{2H-1}}{M}\left|Z_s^n\right|^2+\frac{\left|\varphi(s)\right|^2}{a^2(s)}.\end{aligned}$$

Therefore, for sufficiently large $\delta>0$, choosing δ such that $\left(\delta-5-\frac{M}{s^{2H-1}}\right)>1$, we obtain for $M>1$

$$\begin{aligned}&\mathbb{E}\int_t^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2 ds\\&+\mathbb{E}\int_t^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2 ds\\\leq\;&C\mathbb{E}\left(e^{\delta A(T)}\left|\xi\right|^2+\int_t^T e^{\delta A(s)}\frac{\left|\varphi(s)\right|^2}{a^2(s)}ds\right)\\<\;&\infty.\end{aligned}$$

Finally, we get

$$\begin{aligned}&\mathbb{E}e^{\delta A(t)}\left|Y_t^n\right|^2\\&+\mathbb{E}\int_0^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2 ds\\&+\mathbb{E}\int_0^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2 ds\\\leq\;&C\mathbb{E}\left(e^{\delta A(T)}\left|\xi\right|^2+\int_0^T e^{\delta A(s)}\frac{\left|\varphi(s)\right|^2}{a^2(s)}ds\right)\\<\;&\infty.\end{aligned}$$

Step 2: Convergence result. From (20) and (21), there exists a process Y^n such that $Y_t^n\nearrow Y_t$ a.s. for all $t\in[0,T]$. Therefore, it follows from Fatou's lemma together with the dominated convergence theorem that

$$\left\{\begin{array}{l}\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s\right|^2 ds\leq C,\text{ and}\\\lim_{n\to\infty}\int_0^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n-Y_s\right|^2 ds=0.\end{array}\right.\tag{23}$$

Next, for all $n\geq1$, by Itô's formula applied to $e^{\delta A(t)}\left|Y_t^{n+1}-Y_t^n\right|^2$, we get

$$\begin{aligned}&e^{\delta A(t)}\left|Y_t^{n+1}-Y_t^n\right|^2\\=\;&2\int_t^T e^{\delta A(s)}\left(Y_s^{n+1}-Y_s^n\right)\\&\times(\mathbb{E}'f_{n+1}\left(s,\eta_s,Y_s^{n+1},Z_s^{n+1},\acute{Y}_s^{n+1}\right)\\&-\mathbb{E}'f_n\left(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n\right))ds\\&-2\int_t^T e^{\delta A(s)}\left(Y_s^{n+1}-Y_s^n\right)\left(Z_s^{n+1}-Z_s^n\right)dB_s^H\\&-2\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}\left(Y_s^{n+1}-Y_s^n\right)\left(Z_s^{n+1}-Z_s^n\right)ds\\&-\delta\int_t^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2 ds.\end{aligned}$$

Letting $t=0$, it follows from the uniform linear growth condition on the sequence f_n, (property (ii) in previous lemma), Cauchy-Schawrtz inequality and assumption **(A2)** that

$$\begin{aligned}&\mathbb{E}\left|Y_0^{n+1}-Y_0^n\right|^2\\&+\delta\mathbb{E}\int_0^T e^{\delta A(s)}a^2(s)\left|Y_s^{n+1}-Y_s^n\right|^2 ds\\&+\frac{2}{M}\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^{n+1}-Z_s^n\right|^2 ds\\\leq\;&C\mathbb{E}\int_0^T e^{\delta A(s)}(\frac{\left|\varphi(s)\right|^2}{a^2(s)}\\&+a^2(s)\left(\left|Y_s^{n+1}\right|^2+\left|Y_s^n\right|^2+\left|\eta_s\right|^2\right)\\&+\left(\left|Z_s^{n+1}\right|^2+\left|Z_s^n\right|^2\right)ds)^{\frac{1}{2}}\\&\times\left(\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2 ds\right)^{\frac{1}{2}}.\end{aligned}\tag{24}$$

Therefore, from (20) and assumption $(A3)$, we provide the existence of a constant $\tilde{C}>0$ independent of n such that

$$\begin{aligned}&\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^{n+1}-Z_s^n\right|^2 ds\\\leq\;&\tilde{C}\left(\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Consequently, it follows from (23) that $(Z^n)_{n\geq1}$ is a Cauchy sequence in $\tilde{\mathcal{V}}_{[0,T]}^H$. Then there exists an $\mathcal{F}_t$−jointly measurable process $Z\in\tilde{\mathcal{V}}_{[0,T]}^H$ such that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^n-Z_s\right|^2 ds=0.$$

Step 3: (Y,Z) verifies MF-BSDE driven by fBm (2). Since $(Y^n,Z^n)\to(Y,Z)$ in $\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$, along a subsequence which we still denote (Y^n,Z^n), we get

$$(Y^n,Z^n)\to(Y,Z)\;\;dt\otimes d\mathbb{P}\;a.e.,$$

and there exists $\chi \in \tilde{\mathcal{V}}_{[0,T]}^{H}$ such that for all $n \geq 1$, $|Z^n| < \chi \; dt \otimes d\mathbb{P}$ *a.e.* Therefore, by the previous lemmma, we have

$$f_n\left(t, \eta_t, Y_t^n, Z_t^n, \acute{Y}_t^n\right) \to f\left(t, \eta_t, Y_t, Z_t, \acute{Y}_t\right) \; dt \otimes d\mathbb{P} \text{ a.e.},$$

Moreover, from condition (ii) in the previous lemma and (20), we have

$$\left|f_n\left(t, \eta_t, Y_t^n, Z_t^n, \acute{Y}_t^n\right)\right| \leq \Sigma(t) < \infty \; dt \otimes d\mathbb{P} \text{ a.e.},$$

where

$$\Sigma(t) = \varphi(t) + r(t)\left(\left|Y_t^1\right| + \left|\acute{Y}_t^1\right| + |U_t|\right) + \theta(t)\,|\chi_t|\,.$$

Then it follows from the dominated convergence theorem that

$$\mathbb{E}\int_t^T f_n\left(s, \eta_s, Y_s^n, Z_s^n, \acute{Y}_s^n\right) ds \underset{n\to\infty}{\to} \mathbb{E}\int_t^T f\left(s, \eta_s, Y_s, Z_s, \acute{Y}_s\right) ds.$$

Finally, passing to the limit on both sides of fractional MF-BSDE (19), we get that (Y, Z) is a solution of MF-BSDE (2).

Step 4: Minimal solution. Let $\left(\tilde{Y}, \tilde{Z}\right) \in \tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a} \times \tilde{\mathcal{V}}_{[0,T]}^{H}$ be any solution of fractional MF-BSDE (2) and let us consider for any $n \geq 1$ the fractional MF-BSDE (19) with its unique solution (Y^n, Z^n), which converges to (Y, Z). Since $f_n \leq f$ for all $n \geq 1$, we get by virtue of the comparison theorem that $Y^n \leq \tilde{Y}$ for all $n \geq 1$. Therefore, $Y \leq \tilde{Y}$.

That proves that (Y, Z) is the minimal solution for fractional MF-BSDE (2). □

4 Conclusion

In the first part of this work, we have studied mean field backward stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$ under stochastic Lipschitz condition. In the second part of the paper we establish the existence of minimal solution to the mean field backward stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$ under continuous and stochastic linear growth condition. Motivated by the works of [3, 4, 9, 10, 14, 13], we have proved an existence result to this kind of equations, in which the coefficient f is assumed to be continuous and stochastic linear growth condition, more precisely, we have treated the stochastic Lipschitz case. So our method in continuous and stochastic linear growth condition case is similar techniques developed in [10] with some suitable changes due to the difference between the processes and the spaces. We note that pretty much of the technical difficulties coming from the fractional brownien motion, since B^H with $H > \frac{1}{2}$ is not a semimartingale, we cannot use the classical theory of stochastic calculus.

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