Algorithm 1 Combinational mutation strategy of differential evolution algorithm

Determining of parameters for DE
Initialization Generate the initial population
Assess the fitness for each individual
while Termination condition is not satisfied **do**
  Mutation
  Set $F = 0.5$ then find new population for each mutation strategy
  Crossover
  for rand $< Cr$ do crossover process for each mutation strategy
  Evaluate the boundary constraints for each new individual
  Particular Selection
  Find the best solution for each mutation strategy
end while
Global Selection
Find the best global solution for each mutation strategy
Output Global optimum solution

Table 1: Test problems of an ordinary differential equation.

Problem	Equation	Source	Domain	Exact Solution
ODE1	$y' + x = 0;\ y(0) = 1;\ y(1) = 1/2$	[16]	[0,1]	$y(x) = 1 - x^2/2$
ODE2	$y' + 0.5y - e^{0.8x} = 0;\ y(0) = 2$	[19]	[0,1]	$y(x) = \frac{40}{13}e^{0.8x} - \frac{14}{13}e^{-0.5x}$
ODE3	$y' - (1 + 2x)y^{1/2} = 0;\ y(0) = 1$	[19]	[0,1]	$y(x) = \frac{1}{4}(2 + x + x^2)^2$
ODE4	$y' + 100y - 99e^{2x} = 0;\ y(0) = 0$	[6, 20]	[0,1]	$y(x) = \frac{33}{34}(e^{2x} - e^{-100x})$
ODE5	$y' - 5(y - x)^2e^{5x} - 1 = 0;\ y(0) = 1$	[1]	[0,1]	$y(x) = x - e^{-5x}$
ODE6	$y' + y^3/2 = 0;\ y(0) = 1$	[8, 21]	[0,1]	$y(x) = 1/\sqrt{1 + x}$

Table 2: Test problems of a system ordinary differential equations.

Problem	Equation	Source	Domain	Exact Solution
ODE7	$y_1' - 9y_1 - 24y_2 - 5\cos(x) + \frac{1}{3}sin(x) = 0;\ y_1(0) = \frac{4}{3}$ $y_2' + 24y_1 + 51y_2 + 9\cos x - \frac{1}{3}\sin x = 0;\ y_2(0) = \frac{2}{3}$	[1, 8]	[0,1]	$y_1(x) = \frac{1}{3}\cos x + 2e^{-3x} - e^{-39x}$ $y_2(x) = -\frac{1}{3}\cos x - e^{-3x} + 2e^{-39x}$
ODE8	$y_1' - 32y_1 - 66y_2 - \frac{2}{3}x - \frac{2}{3} = 0;\ y_1(0) = \frac{1}{3}$ $y_2' + 66y_1 + 133y_2 + \frac{1}{3}x + \frac{1}{3} = 0;\ y_2(0) = \frac{1}{3}$	[1, 7]	[0,1]	$y_1(x) = \frac{2}{3}x + \frac{2}{3}e^{-x} - \frac{1}{3}e^{-100x}$ $y_2(x) = -\frac{1}{3}x - \frac{1}{3}e^{-x} + \frac{2}{3}e^{-100x}$

where ϵ is a small number. Stopping criteria (19) have been used in paper [11, 12] and can be made our optimization algorithm stop in the minimum condition that can be reached. To validate how good the performance of the model, the Root of the Mean Squared Error (RMSE) and the Maximum Error (MAXE) are calculated using the numerical solution y_{trial} and the exact solution y, respectively:

$$\text{RMSE} = \sqrt{\frac{1}{n}||\sum_{i=1}^{n}(y_{\text{trial}})_i(t) - y_i(t)||^2} \quad (20)$$

$$\text{MAXE} = \max ||(y_{\text{trial}})_i(t) - y_i(t)|| \quad (21)$$

where n is the total number of collocation points. Note that this error is not considered in the process of solving of ODEs that have no exact solutions so that the approximation result can be achieved from FFV.

3 Results of the Approximation Solutions

Here, we solve several IV problems that have analytical solutions using the CmDE algorithm and optimization model for solving Stiff ODE, in order to validate the capability and the accuracy of our method. The method is implemented to approximate various types of ODEs, like Stiff and non-Stiff. The algorithm uses population size of 200, and maximum number of iterations in these computations is 300-10000. In the first calculation, we implement our optimization method for solving stiff and non-stiff ODEs with variations of maximum number of iterations with constant nat value. In the second calculation, we various of nat from the Fourier-like Series and Eq. 15 or RMSE as stopping criteria. All computations are running with MATLAB R2018a in HP Pavilion Laptop Model 14-dv0067TX that is equipped with processor Intel Core TM i7 with 8 GB ram and 4.70 GHz running Windows 10. Several problems of ODEs, Stiff and non-Stiff, are given in Table 1.

The results of several problems of ODEs in Table 1 and Table 2 are given in Table 6, Table 4 and Table 8. In Table 3, we run CmDE algorithm with various maximum iteration for the same nat for each trial solutions from single ODE in Table 1. Then, in Table 4 and Table 5, we run CmDE algorithm again for one Stiff ordinary differential equation and system of Stiff ordinary differential equation with various nat and Equation (15) or RMSE as stopping criteria. The graph of each problem shows in Fig. 1 until Fig. 8.

Table 3: Result of Problems with variation of nat, variation of RMSE as stopping criteria and maximum iteration is 60000.

Problem	nat of Trial Solution	Iteration of the best solution	RMSE as stopping criteria	RMSE	MAXE
ODE1	3	51	1e-04	5.823e-05	1.604e-04
	4	296	1e-05	7.993e-06	3.220e-05
	5	1340	1e-06	9.473e-07	3.894e-06
ODE2	3	256	1e-04	7.304e-05	1.988e-04
	4	890	1e-05	9.085e-06	1.796e-05
	5	4138	1e-06	9.004e-07	3.456e-06
ODE3	7	423	1e-04	6.386e-05	2.801e-04
	8	904	1e-05	8.188e-06	2.143e-05
	9	885	1e-06	6.782e-07	1.397e-06
ODE4	8	14233	1e-04	9.437e-05	2.490e-04
	9	2230	1e-05	9.836e-06	2.718e-05
	10	2481	1e-06	9.738e-07	2.223e-06
ODE5	6	456	1e-04	6.255e-05	1.559e-04
	7	657	1e-05	8.304e-06	2.115e-05
	8	785	1e-06	7.917e-07	2.491e-06
ODE6	3	245	1e-04	8.281e-05	1.809e-04
	4	902	1e-05	8.230e-06	1.537e-05
	5	3590	1e-06	8.706e-07	2.830e-06

Table 4: Result of Problems with variation of nat and maximum iteration is 10000.

Problem	nat of Trial Solution	Iteration of the best solution	Iteration as stoppin criteria	RMSE	MAXE
ODE1	3	801	>800	1.12e-05	2.30e-05
	4	802	>800	3.13e-07	5.93e-07
	5	803	>800	1.46e-08	3.35e-08
ODE2	5	801	>800	3.36e-05	1.33e-04
	7	801	>800	5.43e-07	1.85e-06
	9	2001	>2000	1.91e-08	8.36e-08
ODE3	3	801	>800	1.49e-05	3.16e-05
	4	810	>800	2.10e-07	4.91e-07
	5	1002	>1000	3.17e-09	1.07e-08
ODE4	4	802	>800	8.25e-05	1.38e-04
	6	1005	>1000	1.93e-06	2.55e-06
	7	2001	>2000	6.89e-08	1.25e-07
ODE5	3	1006	>1000	2.17e-05	6.69e-05
	4	3019	>3000	2.40e-06	9.24e-06
	5	5001	>5000	1.50e-07	4.56e-07
ODE6	5	1002	>1000	1.15e-03	7.87e-03
	7	2006	>2000	9.24e-04	6.41e-03
	10	5006	>5000	6.17e-04	3.51e-03

Table 5: Result of Problems of Stiff System using CmDE with variation of nat and maximum iteration is 10000.

Problem	nat of Trial Solution	Iteration of best fitness	Time average (s)	Equation	RMSE	MAXE
ODE7	3	1001	248.69	Equation1	4.63e-04	7.97e-04
				Equation2	4.63e-04	4.33e-04
	4	2003	425.16	Equation1	1.36e-05	3.53e-05
				Equation2	1.36e-05	1.60e-05
	5	3001	675.38	Equation1	1.54e-06	3.30e-06
				Equation2	1.54e-06	1.53e-06
ODE8	3	802	180.86	Equation1	3.31e-05	6.42e-05
				Equation2	3.31e-05	3.17e-05
	4	2001	466.73	Equation1	5.12e-07	9.33e-07
				Equation2	5.12e-07	4.60e-07

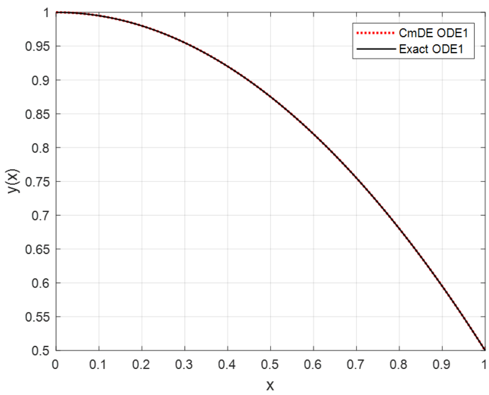

Figure 1: Graph of exact and approximate solution with nat = 5 and RMSE = 9.473e-07 of ODE1

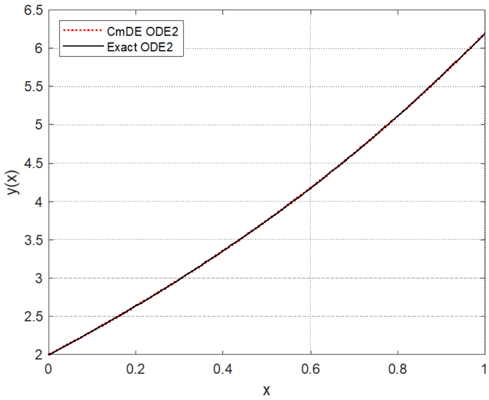

Figure 2: Graph of exact and approximate solution with nat = 5 and RMSE = 9.004e-07 of ODE2

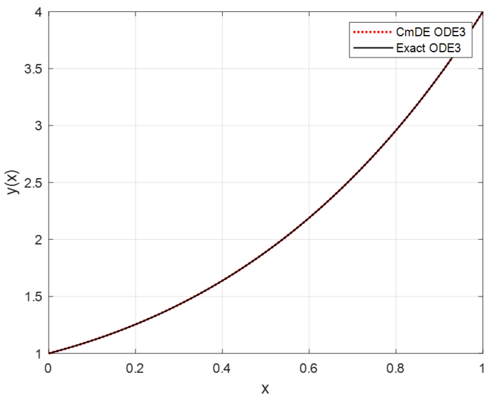

Figure 3: Graph of exact and approximate solution with nat = 9 and RMSE = 6.782e-07 of ODE3

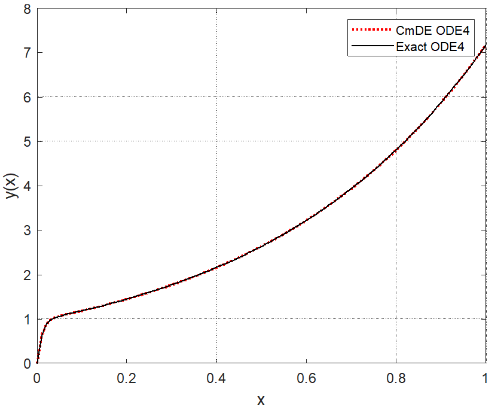

Figure 4: Graph of exact and approximate solution with nat = 7 and RMSE = 9.738e-07 of ODE4

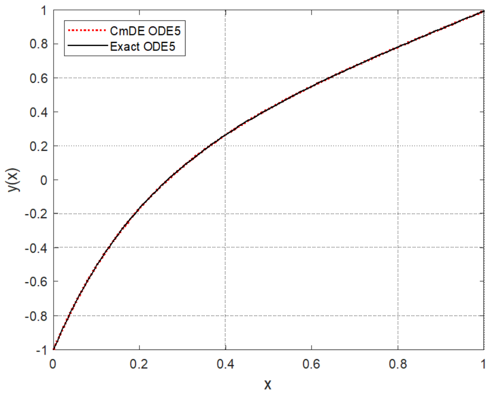

Figure 5: Graph of exact and approximate solution with nat = 8 and RMSE = 7.917e-07 of ODE5

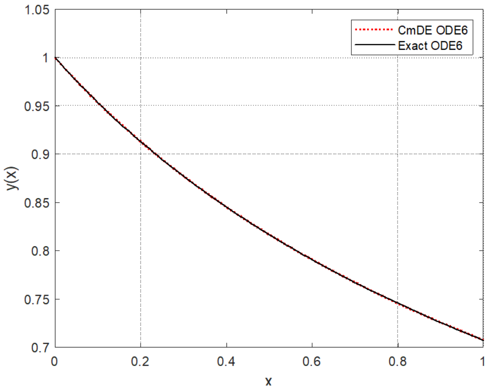

Figure 6: Graph of exact and approximate solution with nat = 5 and RMSE = 8.706e-07 of ODE6

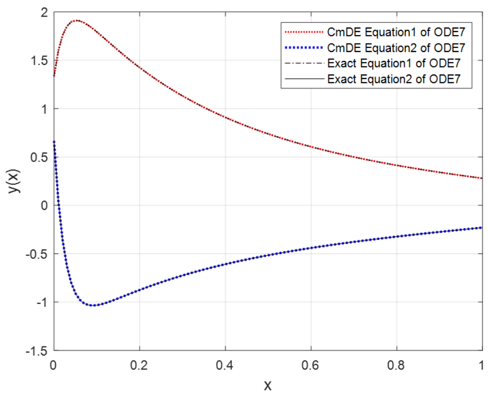

Figure 7: Graph of stiff system of exact and approximate solution with nat = 5 and RMSE1 = RMSE2 = 1.54e-06

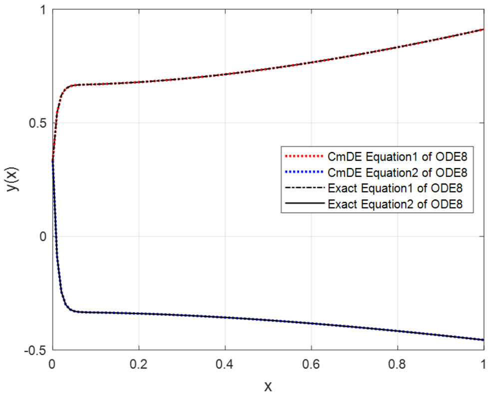

Figure 8: Graph of stiff system of exact and approximate solution with nat = 4 and RMSE1 = RMSE2 = 5.12e-07

4 Discussion

In this research, we obtain numerical solutions for examples of ordinary differential equation especially stiff ordinary differential equation like in Table 1. These ODE problems are changed into an optimization problem using the concepts that we have explain in section 2. Then, optimization from these problems or the solutions for this optimization problems are found using CmDE algorithm. The output for this optimization problem is the coefficients of trial solutions from these problems. In order to analyse accuracy of our methods, in comparison with exact solutions, the RMSE and MAXE is calculated for that ODEs.

In searching the coefficient of trial solution using CmDE algorithm, the setting of maximum iteration be one of factor that can affects the accurate value of the trial solution. From Table 2, we have three different maximum iteration for solving optimization problems with CmDE. We show that if the iteration value become bigger, such that the value of RMSE become smaller. The results show that CmDE algorithm can find approximate solution which can near with the exact solution when the maximum iteration become bigger. Furthermore, a trial solution that we propose can be an approximate solution from ODE that we have shown in Table 1.

Adding iteration in a nat value like in Table 3 can achieve condition where the RMSE value stagnant in certain value. This is caused by the long series (nat) only able to approximate the exact solution of the ordinary differential equation in that nat value. Then, if we want to get the better accuracy, then the nat value has to be increased. In Table 3 and 4, we show the comparison of optimization result for different nat or different of a series sum. Then, to see the accuracy that can be achieved from a nat value, we add Eq. (15) as one of stopping criteria. The result of Table 3 and 4 show that we can increase the accuracy by increasing the value of nat. We can see it from the RMSE value for each problem. The graph of each problem shows in Fig. 1 until Fig. 8. The boundary of independent variable t or x can be extended to the bigger space. Trial solution of each ODE in Table 1 can also apply to larger x limit.

This optimization method can be extended for solving differential equation that does not have an exact solution. The accuracy of the solution of differential equations can be seen from the fitness value. As we purpose in Table 3, the fitness value linearly proportion to the RMSE value. Therefore, for differential equation that does not have the exact value, the fitness value can be used as benchmark from accuracy of the approximate solution. Thus, when the fitness value become smaller, then the approximate solution can nearly approximate the solution of that differential equation.

5 Conclusion

In solving differential equations, we build the approximate solution in a series as trial solution. This trial solution can be used in stiff ordinary differential equation and non-stiff ordinary differential equation as a base approximated function such that solving ODE problem can be transformed into an optimization problem. The aim is to minimize the weighted residual function, which is the error obtained from the implementation of the series into the differential equations. Boundary and initial conditions are imposed as constraints that are implemented as the penalty in the objective function.

We use Combinational mutation strategy of Differential Evolution (CmDE) algorithm as a tool to minimize the residual function. This CmDE algorithm is successfully giving the most minimum results for weighted residual function in the trial solution of each ODE. Thus, metaheuristic algorithms like CmDE algorithm can be applied to approximate solutions of many differential equation problems. This algorithm will give robust tools in a simple way for approximating the complex linear ordinary differential equations. Therefore, we motivate to build a general approximate solution to approximate the nonlinear ODEs that can apply to Stiff differential equation and non-Stiff differential equation.

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